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GEOMETRY WHOSE ELEMENT OF ARC IS A LINEAR DIFFERENTIAL FORM, WITH APPLICATION TO THE STUDY OF MINIMUM DEVELOPABLES.

BY C. L. E. MOORE.

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We are accustomed to think of the element of arc as being defined by the square root of a quadratic differential form. This defines length as a pure number which is always taken as positive. Under this definition the distance between two points cannot have a sign attached unless it is fixed arbitrarily for each direction of the surface. There are however two well known classes of surfaces for which the element of arc is a linear differential form, viz., minimum planes and minimum developables. For the minimum plane the element of arc is an exact differential, while for the minimum developable the element of arc is not an exact differential. These are the only two possibilities for a linear differential. By a proper choice of variable one exact (inexact) differential form can be transformed into any other exact (inexact) form but a form of one kind cannot be transformed into the other kind. The linear form of the arc length shows us first of all that on both these surfaces distance is a directed quantity since the sign depends on the direction of integration. On a minimum plane the length of any closed curve is zero while on the minimum developable this is not necessarily true. These surfaces then become all the more interesting because they differ in such a marked respect from ordinary surfaces.

The element of arc being linear, it is not possible on either of these surfaces to define geodesics as the curves which minimize the integral,

$$\int (X dx + Y dy),$$

for such an integral, by varying the path can take any value whatever. For the study of the geometry on such a surface then it is necessary to find some other means of defining what corresponds to straight lines in the plane. As for the minimum plane, straight lines are already defined by means of linear equations in the parameters ordinarily used or by the intersection of the minimum plane with an ordinary plane. However the ordinary definition of angle breaks down so

that in the study of such a geometry angle has to be defined anew. This was done by Beck.[1] A geometry analogous to that in the minimum plane was discussed by Phillips and Moore [2] and many of the properties of the triangle ecc. derived analytically by Beck were there obtained synthetically. Such a geometry was also mentioned by Wilson and Lewis [3] and the analytic definition of angle given.

Phillips and Moore also discussed a second kind of geometry in which the element of arc is not an exact differential. In this paper I shall discuss these two geometries analytically and apply the results to the study of the geometry on the minimum plane and the minimum developable.

In these two geometries, in order to define distance, and angle, a fundamental line f and a fundamental point F were assumed. In the first kind the line passed through the point and in the second kind it did not. The equality of two distances on the same line was then defined as follows, let A, B, C, D, be four points on a line cutting f in P, then AB = CD if A, D and B, C have the same harmonic point with respect to P. The same definition for equality holds for two segments on parallel lines (lines intersecting on f). For any four points on a line,

$$AB = \lambda CD,$$

where

$$\lambda = (AC \mid DP) - (BC \mid DP).$$

This also gives a way of comparing distances on two parallel lines. From this definition it is easily seen that there is only one point on a line through A which is at a given distance from A. If we then assume that the locus of points equidistant from A is an analytic curve it is easily shown that it is a straight line. Knowing this locus we can now compare distances on any two lines for on any line through A we can mark off a distance AM which has a given ratio to a given distance. As soon then as we have assumed a unit of length we can measure all distances. Distance being defined in terms of double ratios the discussion which follows can be considered as a problem in pure projective geometry. It is to be noted that the above does not give any way of comparing distances on lines through F or on f. The same

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1 Zur Geometrie in der Minimalebene. Archiv der Mathematik und Physik, **20**, (1913).

2 An algebra of plane projective geometry. These Proceedings, **47** (1912). In what follows I shall refer to this paper as A. P. G.

3 The space-time manifold of relativity. The non-euclidean geometry of mechanics and electrodynamics. These Proceedings, **48** (1913).

thing, however, happens in euclidean geometry as there is no way of comparing intervals on minimum lines or on the line at infinity.

Case I.

The fundamental facts of the first kind of geometry are:

Distance is a directed quantity, that is when a positive direction is fixed on one line of the plane it is fixed for each line of the plane.

The locus of points equidistant from a fixed point A is a line passing through F.

The distance between two points A and B is the same as the distance between any point on AF and any point on BF.

The sum of the sides of a triangle is zero.

The distance from a point, not on f, to a point (not F) on f is infinite. The distance from a point not on F is indeterminate. The distance from a point A to any point on AF is zero.

Angle is a directed quantity, that is when a positive rotation is fixed about one point of the plane it is fixed about each point of the plane.

Lines making a fixed angle with a given line a pass through the same point on f.

The angle between two lines a and b is the same as the angle between any line passing through (a, f) and any line passing through (b, f).

The sum of the angles of a triangle is zero.

The angle between a line, not through F, and a line (not f) through F is infinite. The angle between f and a line through F is indeterminate. The angle between two lines intersecting on f is zero.

If a, b, c, denote the sides of a triangle and A, B, C, the angles opposite, the relations connecting them were found to be,

$$\mathrm{A} + \mathrm{B} + \mathrm{C} = 0, \qquad a + b + c = 0,$$

$$\frac{a}{\mathrm{A}} = \frac{b}{\mathrm{B}} = \frac{c}{\mathrm{C}}.$$

Defining the line area of a triangle to be such a scalar function of the sides taken in a definite order around the triangle, that triangles having the same vertex and bases on the same line shall have areas proportional to their bases, the following formula for area was derived,

$$\text{Line area} = ab\mathrm{C} = ac\mathrm{B} = bc\mathrm{A}.$$

A dual definition for angle area led to,

$$\text{Angle area} = \text{AB}c = \text{AC}b = \text{BC}a.$$

Coördinate systems.

The fact that the locus of points equidistant from a given point is a line suggests that one convenient coördinate system would be a point 0 and a line l passing through it. The coördinates of a point A are then the distance 0A and the angle which 0A makes with l. These numbers will fix a point uniquely. The distance between two points (ρ_1, θ_1) and (ρ_2, θ_2) is then,

$$d = \rho_2 - \rho_1.$$

The angle between the two lines is,

$$\theta = \theta_2 - \theta_1.$$

The element of arc is,

$$ds = d\rho.$$

This coördinate system is not suitable for a great many problems, for the coördinates of the points of the line 0F are indeterminate, each point having the coördinates $(0, \infty)$. For this system of coördinates the equation of a straight line is not linear which is another disadvantage.

A second and more convenient coördinate system is obtained by taking the three points 0 (not on f), F, P (on f), as the vertices of a triangle of reference. Number the points along 0P by beginning with 0 and laying off equal distances along 0P. Draw lines from F to these points and number them to correspond. Number the lines through 0 by beginning with 0P and constructing equal angles about 0. Number the points in which these lines cut f to correspond. Finally number the points on 0F by taking an assigned interval 0Q as unity and measuring off equal intervals by choosing Q_1, to satisfy the relation $(f\text{Q} \mid 0\text{Q}_1) = -1$, then choose Q_2 so that $(f\text{Q}_1 \mid 0\text{Q}_2) = -1$ *ecc.* If these relations are satisfied we will say that the intervals 0Q, QQ_1, Q_1Q_2, ecc. are equal. Draw lines from P to these points and number them to correspond. The coördinates of a point will then be the projection of the point on 0F and 0P from P and F. We will call the numbers along 0F, y, and those along 0P, x, then it is seen from the fundamental properties of distance that the distance between (x_1, y_1) and (x_2, y_2) is,

$$d = x_1 - x_2,$$

that is the distance between two points is the same as the distance between the points into which they project from F.

With this system of coördinates the equation $y = kx$, represents a line through 0. For let 0Q be a line passing through 0, Figure 1,

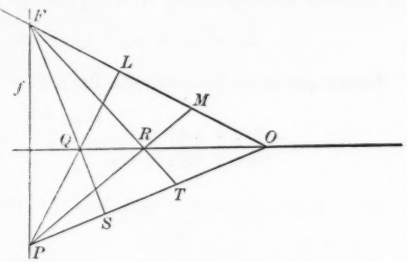

FIGURE 1.

and let (x_1, y_1) be the coördinates of R and (x, y), the coördinates of Q. Then from the properties of distance it is seen at once that, 0Q = 0S, 0R = 0T. From the definition of measure on 0F and the comparison of two distances on any line we see that,

$$0L : 0M = 0Q : 0R$$

Therefore

$$\frac{0L}{0M} = \frac{0S}{0T}, \text{ or } \frac{y}{y_1} = \frac{x}{x_1}, \text{ or } y = \frac{y_1}{x_1} x.$$

In the system of measure established on f we will take the coördinates of the point in which the line $y = kx$ cuts f as k. Then the coefficient k will represent the angle which 0Q makes with 0P.

To find the equation of a line joining two points, we will take as the defining property of the line the fact that the area of the triangle formed by any three of its points is zero. Let $Q_1(x_1, y_1)$, $Q_2(x_2, y_2)$ be two fixed points and Q (x, y) be any point on the line joining them, then

$$QQ_1Q_2 = 0QQ_1 + 0QQ_2 - 0Q_2Q_1.$$

Using the formula for area this becomes

$$\left(\frac{y}{x} - \frac{y_1}{x_1}\right) x x_1 - \left(\frac{y}{x} - \frac{y_2}{x_2}\right) - \left(\frac{y_2}{x_2} - \frac{y_1}{x_1}\right) x_1 x_2 = 0,$$

or
$$(x - x_1)(y_1 - y_2) = (y - y_1)(x_1 - x_2).$$

That is the equation of the straight line is of the first degree.

The coördinates of a line will be the dual of the coördinates of a point and therefore will be the numbers corresponding to the points in which the line cuts f and 0P. The angle between two lines will be the difference between the numbers corresponding to the points in which it cuts f. The number corresponding to the point in which the line,

$$ux + vy = w,$$

cuts f is $-\frac{u}{v}$ Hence the angle between the lines

$$u_1x_1 + v_1y_1 = w_1,$$
$$u_2x_2 + v_2y_2 = w_2,$$

is

$$\theta = \frac{u_1v_2 - u_2v_1}{v_1v_2} \tag{2}$$

Curvature. Using the ordinary definition of curvature we can now derive the expression for the curvature, in this geometry, for a curve

$$y = f(x).$$

The tangent at the point (x_1, y_1) is

$$y - y_1 = f'(x)\,(x - x_1),$$

and the tangent at the near by point $(x_1 + dx, y_1 + dy)$ is,

$$y - y_1 - dy = f'(x_1 + dx)\,(x - x_1 - dx).$$

The angle between the tangents is then

$$f'(x_1 + dx) - f'(x_2).$$

Hence dividing this by the element of arc we have for the curvature,

$$\mathrm{K} = \frac{f(x) + f''(x)\,dx + f'''(x)\,dx^2 + \cdots - f'(x_1)}{dx} = f''(x).$$

From this we see at once that the curves of zero curvature are straight lines. The curves of constant curvature are defined by the differential equation,

$$\frac{d^2y}{dx^2} = k$$

That is the curves of constant curvature are

$$y = \frac{k}{2}x^2 + c_1x + c_2. \tag{4}$$

This curve is then the analog of the circle in the euclidean plane. It

is a conic tangent to the line f at the point F. In the minimum plane this is the curve which Study calls the parabolic circle.

Area. The line area of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3), is

$$\left[\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3}\right](x_1 - x_2)\,(x_3 - x_1) =$$

$$(y_3x_1 - y_1x_3) + (y_1x_2 - y_2x_1) + (y_2x_3 - y_3x_2) =$$

$$\begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix}$$

which agrees with the ordinary formula for area. The angle area is

$$\frac{1}{(x_1 - x_2)\,(x_2 - x_3)\,(x_3 - x_1)} \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix}$$

That is,

Line area = angle area multiplied by the product of the three sides.

The curvature of the parabolic circle circumscribing the above triangle is

$$\frac{2 \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix}}{(x_1 - x_2)\,(x_2 - x_3)\,(x_3 - x_1)} = 2 \times \text{Angle area} = 2 \times \text{line area} \div \text{abc.}$$

This agrees with the curvature of a circle in euclidean geometry except for a factor 2. This is due to the definition of area but there seems to be no need to complicate the formula just to make a closer agreement.

The condition for a point of inflection is the same here as in ordinary geometry,

$$\frac{d^2y}{dx^2} = 0.$$

Collineations of the plane. We saw from the definition of distance that the sides and angles of a triangle are in a measure independent. In fact if the vertices are moved along the lines which join them to the point F the sides are not changed in length while the angles can be made to vary from zero to infinity. We should then expect to find collineations which leave distance invariant and change angle and vice versa. If distance is to be preserved both the point F and the

line f must be left invariant and $x_1 - x_2$ must be transformed into $x^1_1 - x^1_2$. Therefore the collineations which do this are,

$$(5) \qquad \begin{aligned} x^1 &= x + c_1, \\ y^1 &= a_2 x + b_2 y + c_2. \end{aligned}$$

These transformations, as we shall see multiply angle by a constant, therefore there is a four parameter group of transformations which leave distance invariant and multiply angle by a constant. The five parameter group of collineations,

$$(6) \qquad \begin{aligned} x^1 &= b_1 x + c_1, \\ y^1 &= a_2 x + b_2 y + c_2, \end{aligned}$$

is a magnification for both distance and angle. The ratio of magnification for distance is at once seen to be b_1. Since distance and angle are dual conceptions, there is a four parameter group of collineations which leave angle invariant and multiply distance by a constant. We could have taken the equations in angle coördinates and then the transformations leaving angle invariant would have had a form exactly similar to (5).

Similarity transformations do not exist in this plane in the same sense that they exist in the euclidean plane for if the sides of a triangle are left invariant the angles are not necessarily left invariant. If we apply the transformation (5) to the two lines,

$$\begin{aligned} u_1 x + v_1 y &= 0, \\ u_2 x + v_2 y &= 0, \end{aligned}$$

the angle between the transformed lines becomes,

$$\frac{1}{b_2}\left(\frac{u_1}{v_1} - \frac{u_2}{v_2}\right).$$

Hence by these transformations angle is multiplied by $\frac{1}{b_2}$. The collineations which preserve both distance and angle are the transformations of the three parameter group,

$$\begin{aligned} x^1 &= x + c_1, \\ y^1 &= a_2 x + y + c_2. \end{aligned}$$

If we apply the transformation (6) to the above two lines, the angle between the transformed lines becomes,

$$\frac{b_1}{b_2}\left(\frac{u_1}{v_1} - \frac{u_2}{v_2}\right).$$

Then the subgroup of the similarity group which preserves angle are those for which $b_1 = b_2$.

The transformations which preserve distance and reverse angle are

$$x^1 = x + c_1,$$
$$y^1 = a_2 x - y + c_2.$$

Those which preserve angle and reverse distance are,

$$x^1 = -x + c_1,$$
$$y^1 = a_2 x + y + c_2.$$

Those which reverse both distance and angle are,

$$x^1 = -x + c_1,$$
$$y^1 = a_2 x - y + c_2.$$

In this geometry instead of having the ordinary similarity transformations we have two kinds, one which multiplies distance leaving angle invariant and one which multiplies angle leaving distance invariant. There is a three parameter group of motions leaving both invariant. We have also the "umlegung" which reverses distance and the dual which reverses angle.

Since the distance between two points is the same as the distance between any other two points, one on each line joining the two given points to the point F, the question arises, what are the most general analytic transformation of the plane into itself which will preserve distance. In the first place f and F must be invariant. Let the transformation be,

$$x = f(x^1, y^1),$$
$$y = g(x^1, y^1).$$

If it preserves distance,

$$ds = dx = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial y^1} dy^1 = dx^1.$$

Hence,

$$\frac{\partial f}{\partial y^1} = 0,$$
$$\frac{\partial f}{\partial x^1} = 1.$$

Therefore $f(x^1, y^1)$ is independent of y^1 and must have x^1 with coefficient unity, that is,

$$f = x^1 + a,$$

and $g(x^1, y^1)$ is any arbitrary function. We will take g so that in a

given region the transformation will be (1, 1), that is in the angle between two lines through F a branch of a curve will go into a single branch with ends on the lines through F into which the boundaries of the angle transforms.

By such a transformation the curves,

$$u(x + a) + vg(x, y) = w,$$

can be transformed into straight lines. So far then as length is concerned these curves could be taken for straight lines and angle defined accordingly. The whole geometry would then be the same. In this character this geometry differs very much from ordinary geometry.

Case II.

The second kind of geometry discussed in A. P. G. had for fundamental system a line f and a point F not on it. In this geometry angle and distance are so related that any three parts of a triangle determine it. A transformation then which preserves distance must also preserve angle and there is no separation of these groups as in Case I.

The general definitions of distance are the same here as before. There is a metric example here, however, which will serve to make it more concrete. Let the line f be the line at infinity. The distance here defined then becomes equal to the area of the rectangles of which BF is one diagonal and A one vertex. The properties of distance can then be verified in this metrical case.

Distance has the following properties:

AB = AC if BC meets AF on f. That is the locus of points equidistant from a given point is a straight line. This line can be any line of the plane, however, instead of a line through F as in Case I.

The distance from a point A, not on f, to a point P of f is infinite if F does not lie on AP. If F does lie on AP the distance is indeterminate.

The distance from a point A to a point on AF, not on f, is zero.

Distance is a directed quantity, that is if a positive direction is assigned for one line of the plane a positive direction is assigned for each line of the plane. A construction was given for measuring on any line beginning at an assigned point, a distance equal to a given distance.

From these most all properties of distance can be derived.

Angle was defined as the dual of distance and with reference to the same fundamental system. Angle then has the following properties:

$ab = ac$ (where xy denotes the angle between the lines x and y) if the line joining the point of intersection of b and c to F meets a on f, that is the envelope of lines making equal angles with a given line is a point.

The angle which a line a, not through F makes with a line through F is infinite if a does not meet it in f. If this is the case the angle is indeterminate.

Angle is a directed quantity, that is if a positive rotation is assigned around one point of the plane it is assigned around each point of the plane. A construction was given for measuring about any point, beginning at a fixed line, an angle equal to a given angle.

The relations connecting distance and angle are the expressions for the angles of a triangle in terms of the sides. These are,

$$A = \frac{a+b+c}{bc}, \quad B = \frac{a+b+c}{ac}, \quad C = \frac{a+b+c}{ab}.$$

Solved for a, b, c, these relations become,

$$a = \frac{A+B+C}{BC}, \quad b = \frac{A+B+C}{AC}, \quad c = \frac{A+B+C}{AB}.$$

From these can be derived,

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C},$$

which corresponds to the sine law of ordinary trigonometry. It should be kept in mind that distance is a directed quantity and therefore a direction around the triangle must be assumed.

Coördinate systems.

Having these triangle relations it is easy to set up a coördinate system. A very simple one is an ordinary polar system consisting of a point 0 from which distances are measured and a line l, through it from which angles are measured. The coördinates are then the distance from 0 and the direction θ, measured from l. This system does not have the indeterminate character of polar coördinates in the euclidean plane since a point will have just one set of coördinates. It does have the disadvantage, however, of having an indeterminate character for the points of the line 0F. The coördinates of any point on this line are $(0, \infty)$, since the distance from any point to F is zero and the angle which any line makes with a line through F is infinite.

The distance between two points (ρ_1, θ_1), (ρ_2, θ_2) is,

$$d = \rho_2 - \rho_1 + \rho_1 \rho_2 (\theta_2 - \theta_1).$$

For the angle between $0P_2$ and $0P_1$ is $\theta_2 - \theta_1$ and expressing the angle in terms of the sides of a triangle, we have,

$$\theta_2 - \theta_1 = \frac{\rho_1 + d - \rho_2}{-\rho_1 \rho_2}$$

which is the relation above.

The equation of a straight line is,

$$A\rho\theta + B\rho + C = 10,$$

which is obtained by finding the locus of a point at a constant distance from a fixed point. The angle which this line makes with l is,

$$\beta = \frac{1}{A + B},$$

Hence for lines through F, $A = -C$. The element of arc is,

$$ds = d\rho + \rho^2 d\theta.$$

A second and for most purposes more convenient system has two points of reference. Algebraically this is very similar to cartesian coördinates in euclidean geometry but geometrically like bipolar coördinates. Let X and Y be the points of reference so chosen that

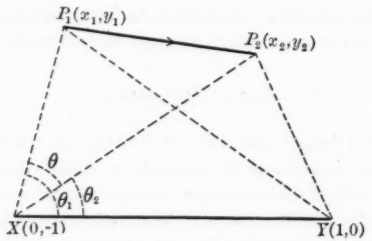

FIGURE 2.

the line XY does not pass through F. The coordinates of a point A is then the distances *from* X, Y *to* the point. The only indeterminate points for this system are those of the line f. For convenience we will choose the distance from X to Y to be unity. The coördinates of X and Y are $(0, -1)$, $(1, 0)$.

The distance between two points (x_1, y_1), (x_2, y_2) is

$$d = x_1 y_2 - x_2 y_1.$$

For in the triangle XP_1Y, fig. 2, from the triangle relations,

$$\theta_1 = \frac{XP_1 + P_1X + YX}{XP_1 + YX} = \frac{x_1 - y_1 - 1}{-x_1}.$$

In the triangle XP_2Y,

$$\theta_2 = \frac{XP_2 + P_2X + YX}{XP_2 \cdot YX} = \frac{x_2 - y_2 - 1}{-x_2}.$$

Then,

$$\theta = \theta_1 - \theta_2 = \frac{y_1x_2 - y_2x_1 + x_2 - x_1}{x_1x_2}.$$

In the triangle XP_1P_2 we have,

$$\theta = \frac{XP_1 + P_1P_2 + P_2X}{XP_1 \cdot P_2X}$$

$$= \frac{x_1 + d - x_2}{-x_1x_2} = \frac{y_1x_2 - x_1y_2 + x_2 - x_1}{x_1x_2}$$

This solved for d gives the value above.

The equation of a straight line in this system of coördinates is linear as can easily be seen by finding the locus of points equidistant from a given point, this we saw was a straight line.

For measuring angles we shall take a third point of reference Z $(1, -1)$. We will then write the equation of the straight line in the form,

$$ux + vy = 1.$$

The equations of ZY, ZX and XY are respectively,

$$x = 1, \qquad y = -1, \qquad x - y = 1.$$

The equation of a line passing through F is

$$y = kx.$$

From the triangle PQZ, fig. 3,

$$\theta = \frac{PQ + QZ + ZP}{PQ \cdot ZP} = \frac{\dfrac{1 + v - u}{uv} + \dfrac{u - v - 1}{v} + \dfrac{v - u + 1}{u}}{\dfrac{v - u + 1}{u} \cdot \dfrac{v - u + 1}{uv}} = u.$$

Similarly we find $\Psi = v$. (It is to be observed that for the angle between two lines, the one is always taken, which does not contain the point F. The other angle is the sum of two infinite angles). The angle between the lines,

$$u_1x + v_1y = 1,$$
$$u_2x + v_2y = 1,$$

is

$$\beta = u_1v_2 - u_2v_1.$$

These formulas for distance and angle show the directed character of both. For the angles of the triangle X Y Z we have,

$$X = 1, \qquad Y = 1, \qquad Z = -1,$$

corresponding to the lengths of the sides opposite,

$$ZY = 1, \qquad XZ = 1, \qquad YX = -1.$$

This triangle will do for triangle of reference for both distance and angle. The angle coördinates of a line are the angles which it makes

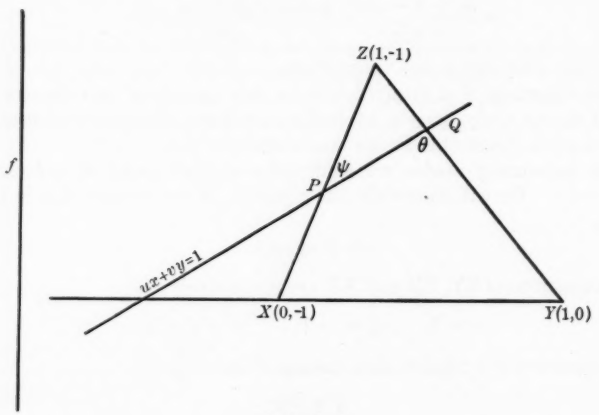

FIGURE 3.

with YZ and XZ. If the equation of a line be written in the form above it will be perfectly dual and the condition that the point (x, y) be on the line (u, v) is precisely,

$$ux + vy = 1.$$

The element of arc in this system of coördinates is,

$$ds = xdy - ydx,$$

and likewise the angle between two nearby tangents of a curve is

$$d\sigma = udv - vdu,$$

where the curve is expressed in angle coördinates.

Using the definition for area as was used in Case I, it was shown in A. P. G. that here also we have two areas:

$$\text{Line area} = a + b + c = \text{A}bc = \text{C}ab = \text{B}ac.$$
$$\text{Angle area} = \text{A} + \text{B} + \text{C} = \text{AB}c = \text{AC}b = \text{BC}a.$$

From these formulas it is at once seen that the line area of any closed curve is equal to its length. That is for a closed curve,

$$\text{Line area} = \text{Length} = \int_c (xdy - ydx).$$

If the curve is given in angle coördinates,

$$\text{Angle area} = \text{Angle sum} = \int_c (udv - vdu).$$

If the equation of a curve is written in the form,

$$y = f(x),$$

the tangent at the point (x_1,y_1) is,

$$xdy - ydx = x_1dy - y_1dx \quad \text{or}$$
$$x\frac{dy}{ds} - y\frac{dx}{ds} = 1$$

If x and y are functions of s the tangent at a nearby point is

$$x\frac{d}{ds}y(s + ds) - y\frac{d}{ds}x(s + ds) = 1$$

or
$$x\left(\frac{dy}{ds} + \frac{d^2y}{ds^2}\cdots\right) - y\left(\frac{dx}{ds} + \frac{d^2x}{ds^2}\cdots\right) = 1.$$

The angle between these two tangents is,

$$d\sigma = \left(\frac{dy}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2y}{ds^2}\right)ds$$

The angle area then becomes,

$$\text{Angle area} = \text{Angle sum} = \int\left(\frac{dy}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2y}{ds^2}\right)ds \tag{7}$$

Curvature. Here as in case I we can define curvature as $\frac{d\sigma}{ds}$. We then have

$$K = \frac{d\sigma}{ds} = \frac{dy}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2y}{ds^2}$$

$$= \left(\frac{dy}{ds}\right)^2 \frac{d}{ds}\left(\frac{\frac{dx}{ds}}{\frac{dy}{ds}}\right) \tag{8}$$

which is the same form as the formula for curvature in euclidean geometry. From (8) we see that curves of zero curvature are straight lines.

If x is taken as the independent variable the differential equation of the curves of constant curvature is,

$$\frac{d^2y}{dx^2} = K\left(x\frac{dy}{dx} - y\right)^3,$$

the solution of which is,

$$c_1y - c_2x = \sqrt{(c_1 - Kx^2)}$$

or

$$(c_1y - c_2x)^2 + Kx^2 = c_1.$$

We will call these curves of constant curvature "pseudo circles." From the equations it is seen that these are conics with respect to which f is the polar of F. The points of a curve for which $K = 0$, are points of inflection. Any collineation of the plane which leave f and F fixed have $K = 0$, as an invariant.

Any conic of the form,

$$A_1x^2 + 2A_2xy + A_3y^2 = 1,$$

is a pseudo circle whose curvature is,

$$K = \frac{1}{4}(A_1A_3 - A^2_2)$$

The line area of a triangle is the sum of the sides. As in Case I, this can be expressed as,

$$\text{Line Area} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$

The area of any closed curve is its perimeter. The dual of this from formula (7), is

$$\text{Angle area} = \int K ds.$$

For curves of constant curvature then,

$$\text{Angle Area} = \text{Line area} \times K.$$

The area of a pseudo circle which does not cut f that is of the curve

$$Ax^2 + 2Bxy + Cy^2 = 1, \quad \text{where} \quad AC - B^2 > 0$$

is

$$\text{Line area} = \int (x dy - y dx) = \int x^2 d\left(\frac{y}{x}\right) = \int_{-\infty}^{\infty} \frac{d\left(\frac{y}{x}\right)}{A + 2B\left(\frac{y}{x}\right) + C\left(\frac{y}{x}\right)^2}$$

$$= \frac{2\pi}{\sqrt{\Delta}}, \qquad \text{where} \quad \Delta = AC - B^2.$$

but $\rho = \frac{1}{K} = \frac{4}{\Delta}$, hence $\rho = \frac{2}{\sqrt{\Delta}}$. The line area of the pseudo circle is $\pi\sqrt{\rho}$. If the pseudo circle cuts the line f the area is infinite.

Collineations of the plane. The general collineation of the plane, which leave F and f invariant is,

$$x^1 = ax + by,$$
$$y^1 = cx + dy.$$

If this transformation be applied to $x_1 y_2 - x_2 y_1$ we have

$$x^1{}_1 y^1{}_2 - x^1{}_2 y^1{}_1 = M(x_1 y_2 - x_2 y_1), \qquad \text{where} \quad M = ad - bc.$$

This is then a magnification for distance. The angle between the lines,

$$u_1 x + v_1 y = 1,$$
$$u_2 x + v_2 y = 1,$$

becomes

$$u^1{}_1 v^1{}_2 - u^1{}_2 v^1{}_1 = \frac{1}{M}(u_1 v_2 - u_2 v_1).$$

The transformation then divides angle. If it is applied to a pseudo circle we have for curvature,

$$K^1 = \frac{1}{M^2} K.$$

If the collineation is a motion

$$ad - bc = 1.$$

These transformations are the area preserving collineations of the plane in euclidean geometry.

If

$$ad = bc = -1,$$

the transformation is an "umlegung," that is a transformation which reverses the sign of distance. For both motion and umlegung the curvature of a curve is an absolute invariant.

In this geometry rotation about a point A is equivalent to a translation along the series of lines passing through the point where the line AF meets f. If the rotation is to be about a finite point that point must remain invariant. But as f and F are left invariant no other finite point can be and hence there are no rotations except identity.

Parallel curves. Parallel curves are defined in ordinary geometry as the envelope of circles of constant radius having their centers on a fixed curve. In this geometry then a parallel to a given curve will be the envelope of the lines which are the loci of points at a constant distance from the points of the curve. The lines corresponding to the same point of the given curve all pass through the same point of f and hence are parallel. That is in a set of parallels to a curve the tangents at corresponding points are parallel. The parallels to the curve

$$f(x, y) = 0,$$

will have for equation the eliminant of

$$f(x_1, y_1) = 0, \qquad xy_1 - yx_1 = a.$$

This however is exactly the process for finding the tangential equation of the curve $f(x, y) = 0$ in euclidean geometry. Hence the parallels to a given curve have the order equal to the class of the original curve and vica versa.

The parallels to the pseudo circle is the eliminant of x_1, y_1 between,

$$Ax^2_1 + 2Bx_1 y_1 + Cy^2_1 = 1,$$

$$xy_1 - yx_1 = a,$$

which is

$$Ax^2 + 2Bxy + Cy^2 = a^2 (AC - B^2).$$

This is again a pseudo circle having the same pair of tangents from F.

Conversely all pseudo circles having the same pair of tangents from F are parallel curves. The form of the equation shows that the parallel is the same whether a is positive or negative. Hence the lines at a distance $\pm a$ from the points of the given curve are parallel tangents to the same parallel curve.

All the lines which cut a given tangent to the curve $f(x, y) = 0$, at a constant angle will pass through the same point. The locus of this point will be the dual of the parallel to a given curve. If the curve is given in angle coördinates,

$$f(u, v) = 0$$

then the angle equation of this locus will have the same degree as the reciprocal of the given curve. The curve which corresponds to a pseudo circle is again a pseudo circle. We again have the same curve whether we take the fixed angle as $\pm a$.

It was shown in A. P. G. that if the line l is the locus of points at a distance k from P then the lines through P all make with l angles equal to $\frac{1}{k}$. An easy calculation will show that the parallel to a pseudo circle corresponding to the distance k and the dual corresponding to the angle $\frac{1}{k}$ are one and the same curve. If we wish then to determine the point of contact of a given generator l of a parallel curve with its envelope all we need to do is to draw a line through the corresponding point of the original curve making an angle $-\frac{1}{k}$ with the tangent line. Where this line cuts l will be the point of contact.

Evolutes. In ordinary geometry the envelope of the normals of a curve is of considerable interest. Aside from the tangent the normal is the only unique line connected with a curve. Here however each line passing through a point of a curve is unique in the same sense, that is it makes a definite angle with the tangent line and there is no other line passing through this point making the same angle. Then connected with every curve there is an infinite number of involutes, that is, the envelope of lines making a fixed angle with the tangent lines. If the original curve is,

$$f(x, y) = 0,$$

the line making the angle k with the curve at the point (x_1, y_1) is

$$(a - ky_1)x + (b + kx_1)y = ax_1 + by_1,$$

where

$$a = \frac{\dfrac{\partial f}{\partial x_1}}{x_1\dfrac{\partial f}{\partial x_1} + y_1\dfrac{\partial f}{\partial y_1}}, \qquad b = \frac{\dfrac{\partial f}{\partial y_1}}{x_1\dfrac{\partial f}{\partial x_1} + y_1\dfrac{\partial f}{\partial y_1}}.$$

The envelope of this line subject to the condition,

$$f(x_1, y_1) = 0,$$

will be the evolute. For the pseudo circle this is again a pseudo circle. In fact if the curve is of the form $f(x, y) = 1$, where $f(x, y)$ is a homogeneous function of x and y, the order of the evolute will be the same as the order of the original curve. The dual of this set of curves is the set traced by the points on the tangent line at a fixed distance from the point of tangency. In the case of a pseudo circle these curves are again pseudo circles.

The pedal curves of a given curve form another interesting configuration. Here for a given point there will be an infinite number of pedals depending on the size of the angle used. If we denote the given curve by C and the dual of the parallel by P and take A as the point with reference to which the pedal is taken a simple construction will show that the pedal curve is the locus of the intersections of the tangents to C with the lines drawn from the corresponding points of P through A. The point A will be a multiple point on the pedal of multiplicity equal to the class of C.

Length preserving transformations of the plane into itself. We found that there was a three papameter group of collineations which left distance invariant. It is evident that there are other transformations which leave distance invariant since the distance between two points measured along various curves may be the same as if measured along straight lines. The transformation,

$$\text{(T)} \qquad x = \mathrm{G}(x^1, y^1), \qquad y = \mathrm{H}(x^1, y^1),$$

will preserve length if,

$$x\,dy - y\,dx = \mathrm{G}\left(\frac{\partial \mathrm{H}}{\partial x^1}\,dx^1 + \frac{\partial \mathrm{H}}{\partial y^1}\,dy^1\right) - \mathrm{H}\left(\frac{\partial \mathrm{G}}{\partial x^1}\,dx^1 + \frac{\partial \mathrm{G}}{\partial y^1}\,dy^1\right)$$
$$= x^1\,dy^1 - y^1\,dx^1$$

from which we see that,

$$G\frac{\partial H}{\partial x^1}q - H\frac{\partial G}{\partial x^1} = -y^1, \tag{9}$$

$$G\frac{\partial H}{\partial y^1} - H\frac{\partial G}{\partial y^1} = x^1.$$

If we make the following simple transformation we can more readily draw conclusions from the above equations.

$$\frac{y}{x} = v, \quad x^2 = u, \quad \frac{y^1}{x^1} = v^1, \quad x^{12} = u^1.$$

Suppose the transformation (T) then becomes,

$$u = f(u^1, v^1), \qquad v = g(u^1, v^1).$$

Then,

$$udv = f\left(\frac{\partial g}{\partial u^1}\, du^1 + \frac{\partial g}{\partial v^1}\, dv^1\right) = u^1\, dv^1,$$

and the relation (9) becomes,

$$f\frac{\partial g}{\partial u^1} = 0, \qquad f\frac{\partial g}{\partial v^1} = u^1.$$

The first equation says that g is a function of v^1 only. Since $\frac{1}{u}$, is an integrating factor and $g\,(v)$ is an arbitrary function of the solution of the differential equation we see from the second equation if written

$$f = u^1 \frac{1}{\frac{\partial g}{\partial v^1}},$$

that $f(u^1, v^1)$ is the reciprocal of an integrating factor. We then have for the original transformation,

$$\frac{H}{G} = F\left(\frac{y^1}{x^1}\right),$$

and $G^2(x^1, y^1)$ is the reciprocal of an integrating factor. The transformation then is such that,

$$G = x^1 f\left(\frac{y^1}{x^1}\right),$$

$$H = g\left(\frac{y^1}{x^1}\right),$$

where f and g are both arbitrary functions of $\frac{y^1}{x^1}$.

By this transformation a curve of the form,

$$u\,\mathrm{G} + v\,\mathrm{H} = 1,$$

will be transformed into a straight line and the theory of distance will be the same if these curves were used for straight lines. There is then an infinite group of transformations of the plane into itself which will preserve distance and consequently will preserve angle. The foregoing is then only one of an infinite number of geometries which can be built up and which are simply isomorphic with the one here discussed.

Application to Minimum Developables.

The equations of a minimum developable can be written in the form,

$$x = \mathrm{R}\left(\frac{u}{v}\right) + v^2\mathrm{R}'\left(\frac{u}{v}\right),$$

$$y = \mathrm{S}\left(\frac{u}{v}\right) + v^2\mathrm{S}'\left(\frac{u}{v}\right),$$

$$z = \mathrm{T}\left(\frac{u}{v}\right) + v^2\mathrm{T}'\left(\frac{u}{v}\right),$$

where $$\mathrm{R}'^2 + \mathrm{S}'^2 + \mathrm{T}'^2 = 0.$$

Primes denote differentiation with respect to $\frac{u}{v}$. Then

$$ds = \sqrt{\mathrm{R}''^2 + \mathrm{S}''^2 + \mathrm{T}''^2}\,(udv - vdu).$$

By a proper choice of variable the expression under the radical can be made equal to unity. In fact the change of variable is,

$$\frac{u^1}{v^1} = \int \frac{d\left(\frac{u}{v}\right)}{(\mathrm{R}''^2 + \mathrm{S}''^2 + \mathrm{T}''^2)^{\frac{1}{2}}}.$$

With this choice of variable we have,

$$ds = udv - vdu.$$

Since this form is the same for all minimum developables it follows that all such surfaces can be mapped on any one of them in such a

way that length will be preserved and in fact this can be done in an infinite number of ways.[4] If we put,

$$u = x, \qquad v = y.$$

we will have the minimum developable mapped on the plane of A. P. G. in such a way that distance is preserved. In this depiction the generators of the developable are carried into the lines through F: the imaginary circle at infinity into the line f. The cuspidal edge will be transformed into the point F. This then differs from the development of the ordinary developable on the euclidean plane since a whole curve is transformed into a point. The transformation, however, does everywhere preserve length.

For lines analogous to geodesics on a minimum developable then we can take the lines which by this depiction go into straight lines. As we saw before this depiction can be made in an infinte number of ways and therefore on a minimum developable there are an infinite number of simply isomorphic geometries. The curves which we shall take as pseudo geodesics are,

$$au + bv = 1.$$

The distance between two points (u_1, v_1), (u_2, v_2) measured on one of these lines is

$$d = u_1 v_2 - u_2 v_1.$$

The angle between two lines

$$a_1 u + b_1 v = 1,$$
$$a_2 u + b_2 v = 1,$$

can then be defined as,

$$\beta = a_1 b_2 - a_2 b_1,$$

and will then be the exact dual of the distance between two points. The area of the triangle (u_1, v_1), (u_2, v_2), (u_3, v_3) will be

$$A = \begin{vmatrix} 1 & u_1 & v_1 \\ 1 & u_2 & v_2. \\ 1 & u_3 & v_3 \end{vmatrix}$$

[4] From the ordinary formula the curvature of a minimum developable i indeterminate. However they can all be mapped on a minimum cone (point sphere). If the curvature of a sphere be taken as the reciprocal of the radius, the curvature of the point sphere is infinite. If then we say that applicable surfaces have the same curvature, the curvature of a minimum developable is infinite.

A pseudo geodesic curvature of a curve can be defined as for the plane and will be found to be,

$$K = \frac{du}{ds}\frac{d^2v}{ds^2} - \frac{dv}{ds}\frac{d^2u}{ds^2},$$

where the curve is defined parametrically on terms of arc length. The pseudo geodesic circle of constant radius will be,

$$au + bv = 1,$$

And the pseudo geodesic circle of constant curvature will be

$$Au^2 + 2Buv + Cv^2 = 1,$$

and the curvature is,

$$K = \frac{AC - B^2}{4}.$$